Lecture 11

2020 A
Fall, 2020

- Pf of changes of variables formulas
- Examples.

Recall the following facts:

<u>Fact I</u> (<u>Inverse</u> <u>Function</u> <u>theorem</u>) D <u>region</u> in $\mathbb{R}^2$

$\Phi: D \longrightarrow \mathbb{R}^2$ C^1-<u>map</u>, $(u_0, v_0) \in$ <u>interior of</u> D

$\Phi(u_0 v_0) = (x_0, y_0)$.

If $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ at (u_0, v_0), $\exists\, D_1 \subset D$ containing (u_0, v_0) in its interior, D_2 containing (x_0, v_0) in its interior, such that $\Phi|_{D_1}$ maps D_1 1-1 onto D_2 with C^1-inverse.

<u>Fact II</u> <u>Let</u> $\underset{(s,t)}{D_1} \xrightarrow{\Phi_1} \underset{(u,v)}{D_2} \xrightarrow{\Phi_2} \underset{(x,y)}{D_3}$ Φ_1, Φ_2 C^1-maps.

<u>Let</u> $\Phi = \Phi_2 \circ \Phi_1$. <u>then</u>

$$J_\Phi = J_{\Phi_2} J_{\Phi_1},$$ and

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(s,t)}$$

e.g.

$$\underset{(s,t)}{D_1} \longrightarrow \underset{(u,v)}{D_2} \longrightarrow \underset{(x,y)}{D_3}$$

$$\begin{cases} u = \sin t \\ v = ts \end{cases}, \quad \begin{cases} x = u + v \\ y = uv. \end{cases}$$

then $\begin{cases} x = \sin t + ts \\ y = \sin t \times ts. \end{cases}$

we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1 & 1 \\ v & u \end{pmatrix}, \quad \frac{\partial(u,v)}{\partial(s,t)} = \begin{pmatrix} \cos t & 0 \\ s & t \end{pmatrix},$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \cos t + s & t \\ \sin t + ts\cos t & t\sin t \end{pmatrix}.$$

can verify $\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(s,t)}$.

Fact III $\Phi : D \to \mathbb{R}^2$ C^1-map, 1-1 onto with C^1-inverse. then

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \text{ in } D$$

Pf : $\Phi^{-1} \circ \Phi = id \iff \Phi^{-1} \circ \Phi(u,v) = (u,v)$, From Fact II

$$1 = \frac{\partial(u,v)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} \neq 0.$$

Theorem 1 (Change of Variables formula)

$\Phi : \tilde{D} \longrightarrow D$, 1-1 onto C^1-map (C^1-inverse)

For F conti in D,

$$(\star) \quad \iint_D F(x,y)\, dA(x,y) = \iint_{\tilde{D}} F \circ \Phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v).$$

Remark. theorem still holds when Φ fails to be 1-1 on several points or curves (as the integration over these small parts are zero).

e.g. $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$,

$\tilde{D} \subset [0,\infty) \times [0, 2\pi)$

$$J_\Phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \cos\theta \, r\cos\theta - (-r\sin\theta)\sin\theta$$
$$= r.$$

(☆) reduces to the old formula

$$\iint_D F(x,y)\, dA(x,y) = \iint_{\tilde{D}} F(r\cos\theta, r\sin\theta) \underbrace{r\,dr\,d\theta}_{dA(r,\theta)}.$$

Fact IV Let $D_1 \xrightarrow{\Phi_1} D_2 \xrightarrow{\Phi_2} D_3$ and $\Phi = \Phi_2 \circ \Phi_1$. If (☆) holds for Φ_1 and Φ_2, then (☆) holds for Φ too.

Pf: We have, for $\underset{\text{conti}}{G : D_2 \to \mathbb{R}}$, $\underset{\text{conti}}{F : D_3 \to \mathbb{R}}$, (☆) means

$$\iint_{D_2} G(u,v)\,dA(u,v) = \iint_{D_1} (G\circ\Phi_1)(s,t)\left|\frac{\partial(u,v)}{\partial(s,t)}\right| dA(s,t), \quad ①$$

$$\iint_{D_3} F(x,y)\,dA(x,y) = \iint_{D_2} (F\circ\Phi_2)(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v). \quad ②$$

Taking
$$G(u,v) = F\circ\Phi_2(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \quad \text{in } ①,$$

$$\iint_{D_2} (F\circ\Phi_2)(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v)$$

$$= \iint_{D_1} (F\circ\Phi_2\circ\Phi_1)(s,t)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\left|\frac{\partial(u,v)}{\partial(s,t)}\right| dA(s,t)$$

$$= \iint_{D_1} (F\circ\Phi)(s,t)\left|\frac{\partial(x,y)}{\partial(s,t)}\right| dA(s,t) \quad \text{(Fact II)}$$

But, by ②, the LHS is $\iint_{D_3} F(x,y)\,dA(x,y)$, done. #

<u>Proof of Theorem 1</u>. For simplicity let's take $D_1 = R = [a,b]\times[c,d]$. Writing

$$\Phi(u,v) = (f_1(u,v), f_2(u,v)),$$

$$J_{\Phi} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u};$$

Clearly, as $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ by assumption, $\frac{\partial f_1}{\partial u}$ and $\frac{\partial f_1}{\partial v}$ can't vanish simultaneously. Fix a partition P on R so refined that on each subrectangle R_j, either $\frac{\partial f_1}{\partial u} \neq 0$ or $\frac{\partial f_1}{\partial v} \neq 0$.

Let

$$\mathcal{A} = \left\{ R_j : \frac{\partial f_1}{\partial u} \neq 0 \text{ on } R_j \right\},$$

$$\mathcal{B} = \left\{ R_j : \frac{\partial f_1}{\partial v} \neq 0 \text{ on } R_j \right\}.$$

Using

$$\iint_R F \circ \Phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$$

$$= \sum_{R_j \in \mathcal{A}} \iint_{R_j} (F \circ \Phi)(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v) +$$

$$\sum_{R_j \in \mathcal{B}} \iint_{R_j} (F \circ \Phi)(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v),$$

if we can show that ($\mathcal{A}$) holds on each R_j, it holds for the general R.

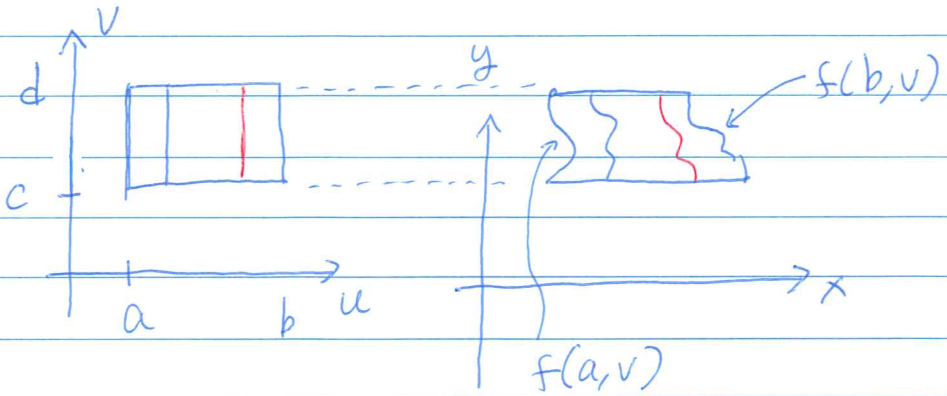
Fact V: ($\star$) holds when Φ is of either one of the following forms:

(i) $\Phi : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} f(u,v) \\ v \end{pmatrix}$, $\frac{\partial f}{\partial u} \neq 0$ on D, or

(ii) $\Phi : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u \\ g(u,v) \end{pmatrix}$, $\frac{\partial g}{\partial v} \neq 0$ on D.

Pf: We consider (i) only. The treatment for (ii) is similar.

First assume $\frac{\partial f}{\partial u} > 0$ on $D = R = [a,b] \times [c,d]$.



The vertical line segment (a,v), $v \in [c,d]$, maps to $(f(a,v),v)$

- - - - - - - - - - - - (b,v), $v \in [c,d]$, - - - - $(f(b,v),v)$

(u,v), $v \in [c,d]$ - - - - $(f(u,v),v)$

As $\frac{\partial f}{\partial u} > 0$, all these curves do not intersect each other.

$$\Phi(R) = \{(x,y) : c \le y \le d,\ f(a,y) \le f(x,y) \le f(b,y)\}$$

Fubini thm:

$$\iint_{\Phi(R)} F(x,y)\, dA(x,y) = \int_c^d \left(\int_{f(a,y)}^{f(b,y)} F(x,y)\, dx \right) dy$$

Use the 1-dim change of variables : $x = f(u,y)$ (y fixed)

$a \mapsto f(a,y)$

$b \mapsto f(b,y)$

$$= \int_c^d \left(\int_a^b F(f(u,y),y) \frac{\partial f}{\partial u} du \right) dy$$

$$= \int_c^d \left(\int_a^b F(f(u,v),v) \frac{\partial f}{\partial u} du \right) dv$$ (change y back to v)

$$= \iint_R F \circ \Phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv, \quad \text{done.}$$

Note that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ 0 & 1 \end{bmatrix} = \frac{\partial f}{\partial u} > 0 .$$

when $\frac{\partial f}{\partial u} < 0$,

$$\Phi(R) = \{ (x,y) : c \le y \le d,\ f(b,y) \le f(x,y) \le f(a,y) \}$$

$$\iint_{\Phi(R)} F(x,y)\, dA(x,y) = \int_c^d \left(\int_{f(b,y)}^{f(a,y)} F(x,y)\, dx \right) dy$$

now $x = f(u,y)$ as before,

$$= \int_c^d \left(\int_b^a F(f(u,y),y) \frac{\partial f}{\partial u} du \right) dy$$

$$= \int_c^d \left(\int_b^a F(f(u,v),v) \frac{\partial f}{\partial u} du \right) dv$$

$$\int_c^d \left(\int_a^b F(f(u,v),v) \left| \frac{\partial f}{\partial u} \right| du \right) dv$$

$$= \iint_R F \circ \Phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv, \quad \text{done } \#$$

<u>Fact VI</u> Factorization of Φ.

For $(u,v) \in R_j \in \mathcal{A}$, let

$$\Phi_1 : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} f_1(u,v) \\ v \end{pmatrix},$$

$$\Phi_2 : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ h(s,t) \end{pmatrix}$$

If we take $h = f_2 \circ \Phi_1^{-1}$, then

$$\Phi = \Phi_2 \circ \Phi_1 .$$

$$\begin{tikzcd} R_j \arrow[rr, "\Phi"] \arrow[dr, "\Phi_1"'] & & D_j \\ & Q_j \arrow[ur, "\Phi_2"'] & \end{tikzcd}$$

Here use Fact I to show Φ_1^{-1} exists and C^1.

Pf = A direct check

$$\Phi_2(\Phi_1(u,v))$$

$$= \Phi_2(f_1(u,v), v)$$

$$= (f_1(u,v), h(f_1(u,v), v))$$

$$= (f_1(u,v), h(\Phi_1(u,v)))$$

so, $$= (f_1(u,v), f_2(u,v))$$

$$= \Phi(u,v)$$

if $f_2(u,v) = h \circ \Phi_1(u,v)$, ie, $h = f_2 \circ \Phi_1^{-1}$.

Similarly, we have factorization for $R_j \in \mathcal{B}$.

Combining Fact V and Fact IV, ($\star$) holds for R.

Once again let me summarize the "train of thoughts" in this proof.

(1) Any Φ can be factorized into $\Phi = \Phi_2 \circ \Phi_1$ where

Φ_i, $i=1,2$, are of a special form.

(2) prove that ($\star$) holds for maps of this special form.

(3) Use Fact II: ($\star$) holds for $\Phi_1, \Phi_1 \Rightarrow$ holds for $\Phi_2 \circ \Phi_1$.

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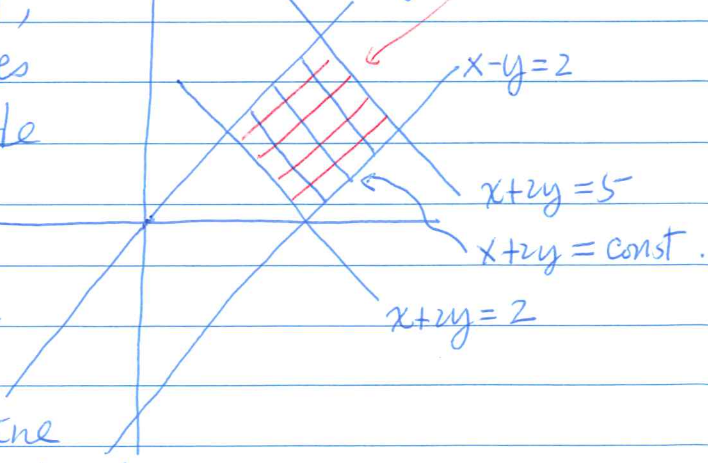
Some examples.

e.g. 1 $\displaystyle\iint_P F(x,y)\,dA(x,y)$ where P is the parallelogram formed by

$x-y=0$, $x-y=2$,

$x+2y=2$, $x+2y=5$

The parallelogram is foliated by the curves $x-y=u$, $u\in[0,2]$, As c runs from 0 to 2, the curves (red lines) runs from the upper side to the lower side. On the other hand, the curves $x+2y=v$, $v\in[2,5]$ runs from the downside to the upper side.



Each red line cuts a blue line at one and only one point which indicates $\binom{x}{y} \mapsto \binom{u}{v}$ is 1-1 onto. So we try (u,v) as the new variables.

$$\begin{cases} u = x - y \\ v = x + 2y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}(2u+v) \\ y = \frac{1}{3}(-u+v) \end{cases} \leftarrow \Phi$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = \frac{1}{3}$$

$$\tilde{P} = [0,2]\times[2,5]$$

$$\therefore \iint_P F(x,y)\, dA(x,y) = \iint_{\tilde{P}} F\left(\tfrac{1}{3}(2u+v), \tfrac{1}{3}(-u+v)\right)\tfrac{1}{3}\, dA(u,v)$$

$$= \frac{1}{3}\int_0^2\int_2^5 F\left(\tfrac{1}{3}(2u+v), \tfrac{1}{3}(-u+v)\right) du\, dv.$$

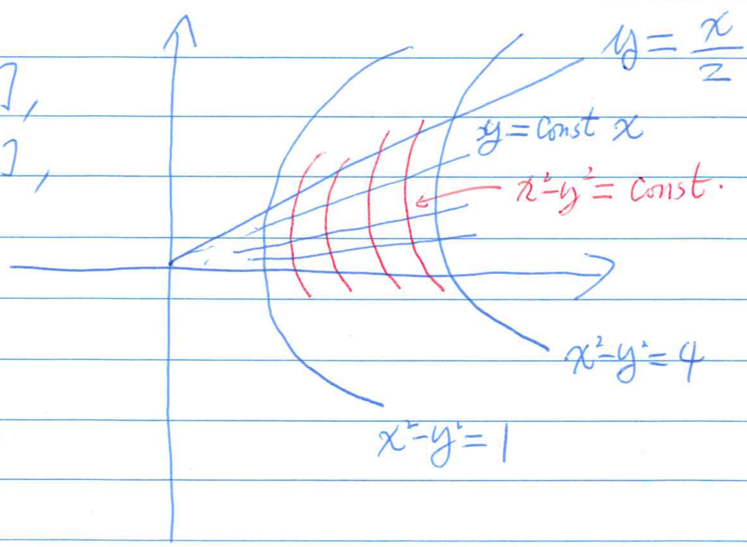
the region $\tilde{P}$ is easier to handle than P.

e.g. 2 $\iint_D \frac{y}{x}\, dA(x,y)$ where D is bounded by
$y = \frac{x}{2}$, $y = 0$, $x^2-y^2=4$, $x^2-y^2=1$

Let (u,v), $u\in[0,\frac{1}{2}]$,
$v\in[1,4]$,

$$\begin{cases} y = ux \\ x^2-y^2 = v \end{cases}$$

ie $\begin{cases} u = y/x \\ v = x^2-y^2 \end{cases} \leftarrow \Phi^{-1}$



Φ is more complicated, so we try to work on Φ^{-1}.

Using the fact II :

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{-2+2\frac{x^2}{y^2}} = \frac{1}{2(u^2-1)}$$

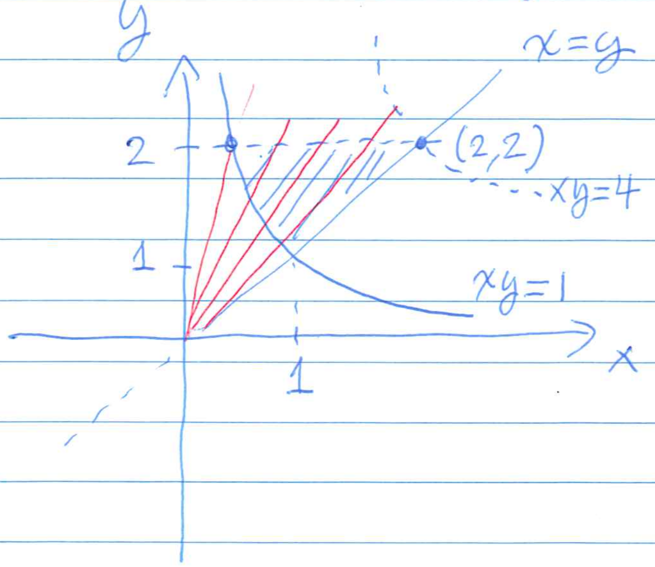
$$\therefore \iint_D \frac{y}{x}\, dA(x,y) = \iint_{\tilde{D}} u \frac{1}{2(u^2-1)}\, dA(u,v)$$

$$= \int_0^{\frac{1}{2}} \int_1^4 \frac{u\, dv\, du}{2(u^2-1)}$$

$$= \frac{3}{2}\int_0^{\frac{1}{2}} \frac{u\, du}{u^2-1}$$

$$= -\frac{3}{4}\log\frac{3}{4} \quad \#$$

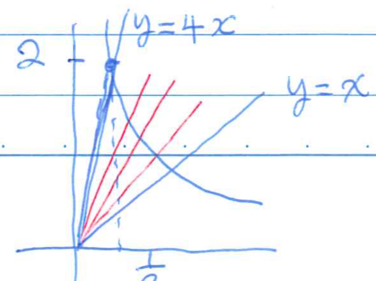
e.g. 3 Evaluate $\displaystyle\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$.

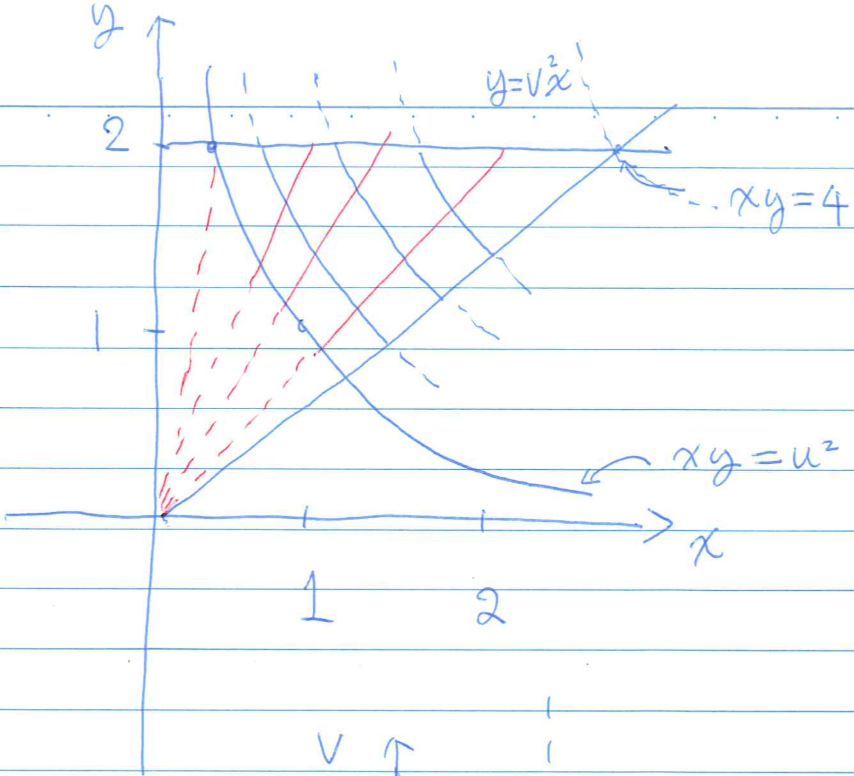


Try $u = \sqrt{xy} \in [1,2]$

$v = \sqrt{\frac{y}{x}}$

$y = v^2 x$, the range should be from 1 to 2.





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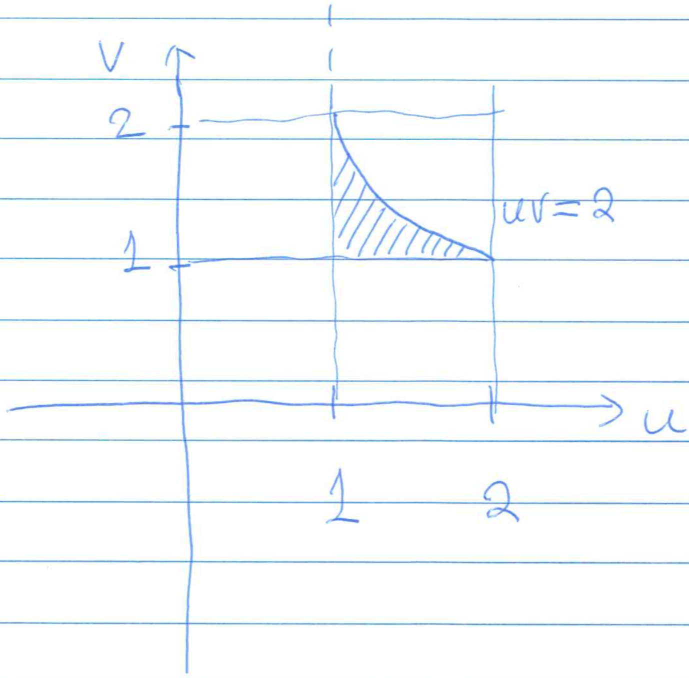


$\tilde{D}$ not a rectangle

$u=\sqrt{xy}\in[1,2]$

$y=2$ goes over to $uv=2$.

$$\begin{cases} u=\sqrt{xy} \\ v=\sqrt{\frac{y}{x}} \end{cases} \Longleftrightarrow \begin{cases} x=\frac{u}{v} \\ y=uv \end{cases}$$

$$J_\Phi = \begin{pmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{pmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{2u}{v}$$

$$\therefore I = \iint_{\tilde{D}} v e^u \frac{2u}{v} \, dA(u,v) = \int_1^2 \int_1^{\frac{2}{u}} e^u 2u \, dv \, du$$

$$= \int_1^2 2ue^u \left(\int_1^{\frac{2}{u}} dv \right) du$$

$$= \int_1^2 (4e^u - 2ue^u)\, du$$

$$\vdots$$

$$= 2e(e-2) \quad \#$$

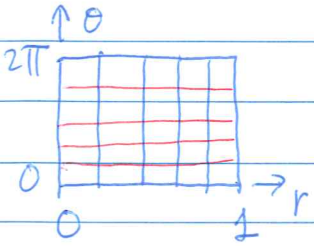
e.g 4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

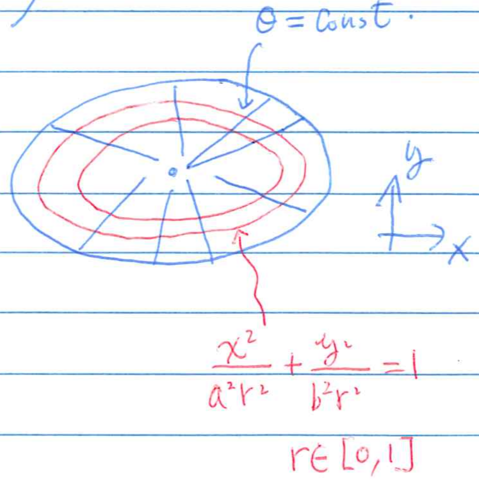
Introduce "generalized polar coordinates"

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$$

$$J_\Phi = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{pmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = abr \geq 0$$





$$\therefore \iint_D 1\, dA(x,y) = \iint_{\tilde{D}} abr\, dA(r,\theta)$$

$$= \int_0^{2\pi} \int_0^1 abr\, dr\, d\theta = \pi ab \quad \#$$

Note: not 1-1 at (a,0),
also at 0, 2π.

The principles in the introduction of (u,v) are,

as we have seen in the above examples, are

① to make the region of integration as simple as possible. Usually, it is a rectangle, sometimes things like

$$\{(u,v) : u \in [a,b],\ f_1(u) \leq v \leq f_2(u)\}.$$

② Need to make sure the integrands do mess up

Sometimes, while the region becomes nice, the integrand

$$F(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

becomes so complicated that it can't be integrated out!

A compromise between the region and the integrand is always needed for a successful computation.